

## A Modular Construction of Unramified $p$ -Extensions of $\mathbf{Q}(\mu_p)$

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### §1. Introduction

An odd prime  $p$  is called irregular if the class number of the field  $\mathbf{Q}(\mu_p)$  is divisible by  $p$  ( $\mu_p$  being, as usual, the group of  $p$ -th roots of unity). According to Kummer's criterion,  $p$  is irregular if and only if there exists an even integer  $k$  with  $2 \leq k \leq p-3$  such that  $p$  divides (the numerator of) the  $k$ -th Bernoulli number  $B_k$ , given by the expansion

$$\frac{t}{e^t-1} + \frac{t}{2} - 1 = \sum_{n \geq 2} \frac{B_n}{n!} t^n.$$

The purpose of this paper is to strengthen Kummer's criterion.

Let  $A$  be the ideal class group of  $\mathbf{Q}(\mu_p)$ , and let  $C$  be the  $\mathbf{F}_p$ -vector space  $A/A^p$ . The Galois group  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$  acts on  $C$  through its quotient  $\Delta = \text{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ . Since all characters of  $\Delta$  with values in  $\bar{\mathbf{F}}_p^*$  are powers of the standard character

$$\chi: \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \Delta \xrightarrow{\sim} \mathbf{F}_p^*$$

giving the action of  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$  on  $\mu_p$ , the vector space  $C$  has a canonical decomposition

$$C = \bigoplus_{i \bmod (p-1)} C(\chi^i),$$

where

$$C(\chi^i) = \{c \in C \mid \sigma c = \chi^i(\sigma) c \text{ for all } \sigma \in \Delta\}.$$

(1.1) **Main Theorem.** *Let  $k$  be even,  $2 \leq k \leq p-3$ . Then  $p \mid B_k$  if and only if  $C(\chi^{1-k}) \neq 0$ .*

In fact, the statement that  $C(\chi^{1-k}) \neq 0$  implies  $p \mid B_k$  is well known [8, Th. 3]. Its converse is also familiar as a consequence of the conjecture that  $p$  is prime to the class number of the real subfield  $\mathbf{Q}(\mu_p)^+$  of  $\mathbf{Q}(\mu_p)$  [8, p. 434]. Thus the con-

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tribution of this paper is to prove that  $p|B_k$  implies  $C(\chi^{1-k}) \neq 0$  without making a supplementary hypothesis.

By a “functoriality” formula for the Artin symbol [20, Th. 11.5, p. 199], this implication is equivalent to

(1.2) **Theorem.** *Suppose  $p|B_k$ . Then there exists a Galois extension  $E/\mathbf{Q}$  containing  $\mathbf{Q}(\mu_p)$  with the following properties:*

$$H \left( \begin{array}{c} E \\ | \\ \mathbf{Q}(\mu_p) \\ | \\ \mathbf{Q} \end{array} \right) G$$

- (a) *The extension  $E/\mathbf{Q}(\mu_p)$  is unramified.*
- (b) *The group  $H$  is a non-zero abelian group of type  $(p, \dots, p)$ , i.e., killed by  $p$ .*
- (c) *If  $\sigma \in G$  and  $\tau \in H$ , then*

$$\sigma \tau \sigma^{-1} = \chi(\sigma)^{1-k} \cdot \tau.$$

In fact, we shall prove (1.2) with  $\mathbf{Q}(\mu_p)$  replaced by the unique subfield  $\mathbf{Q}(\mu_p^{\otimes(1-k)})$  of  $\mathbf{Q}(\mu_p)$  whose degree over  $\mathbf{Q}$  is  $(p-1)/(p-1, k-1)$ . This subfield is the field corresponding to the kernel in  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$  of  $\chi^{1-k}$ .

(1.3) **Theorem.** *Suppose  $p|B_k$ . Then there exists a finite field  $\mathbf{F} \cong \mathbf{F}_p$  and a continuous representation*

$$\bar{\rho}: \text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{GL}(2, \mathbf{F})$$

with the properties:

- (i)  *$\bar{\rho}$  is unramified at all primes  $l \neq p$ .*
- (ii) *The representation  $\bar{\rho}$  is reducible (over  $\mathbf{F}$ ) in such a way that  $\bar{\rho}$  is isomorphic to a representation of the form*

$$\begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix}.$$

That is,  $\bar{\rho}$  is an extension of the 1-dimensional representation with character  $\chi^{k-1}$  by the trivial 1-dimensional representation.

(iii) *The image of  $\bar{\rho}$  has order divisible by  $p$ . In other words,  $\bar{\rho}$  is not diagonalizable.*

(iv) *Let  $D$  be a decomposition group for  $p$  in  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ . Then  $\bar{\rho}(D)$  has order prime to  $p$ , i.e.,  $\bar{\rho}|_D$  is diagonalizable.*

Notice that (1.3) implies (1.2). Indeed, if  $\bar{\rho}$  satisfies the above properties, then the image of  $\bar{\rho}$  is the Galois group of an extension  $E/\mathbf{Q}$  such that  $E$  is of type  $(p, \dots, p)$  over the field  $\mathbf{Q}(\mu_p^{\otimes(1-k)})$ . Now  $E/\mathbf{Q}$  is unramified outside  $p$  by (i), and the  $(p, \dots, p)$  layer is a non-trivial extension by (iii). This  $(p, \dots, p)$  extension is unramified at (the unique prime over)  $p$  by (iv); hence it is *everywhere* unramified. Finally, the

conjugation formula (c) of (1.2) follows from the matrix identity

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ad^{-1}x \\ 0 & 1 \end{pmatrix}.$$

In proving (1.3) we begin by “finding”  $\bar{\rho}$  in the  $p$ -adic representation associated with the modular variety  $J_1(p)$  attached to forms of weight 2 on  $\Gamma_1(p)$ . Assuming that  $p|B_k$ , we construct a normalized eigenform  $f = \sum a_n q^n$  in the space of such cusp forms which satisfies

$$a_l \equiv 1 + l^{k-1} \pmod{\mathcal{M}}$$

for all primes  $l \neq p$ , where  $\mathcal{M}$  is a certain fixed ideal over  $p$  in the field generated by the coefficients  $a_n$ . This leads to our  $\bar{\rho}$ , and by the time we have constructed  $\bar{\rho}$  we know from the construction that (i), (ii), and (iii) of (1.3) are satisfied by  $\bar{\rho}$ . It then remains to prove (iv). We then use the theorem of Deligne-Rapoport that the variety  $J_1(p)/J_0(p)$  acquires everywhere good reduction over the real subfield  $\mathbf{Q}(\mu_p)^+$  of  $\mathbf{Q}(\mu_p)$  [5]. This implies that, locally at  $p$ ,  $\bar{\rho}|_{\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\mu_p)^+)}$  is the representation attached to a finite flat commutative group scheme of type  $(p, \dots, p)$  over the integer ring of the completion  $\mathbf{Q}(\mu_p)^+ \otimes \mathbf{Q}_p$ . We note especially that the absolute ramification index of this completion is  $(p-1)/2 < p-1$ ; this enables us to prove (iv) by applying results of Raynaud [15] on group schemes of type  $(p, \dots, p)$ .

Our proof is motivated by two key ideas of Serre. The first idea (cf. [16]) is that the divisibility of  $B_k$  by  $p$  implies a congruence similar to the above one for some cusp form of weight  $k$  on  $\text{SL}(2, \mathbf{Z})$ ; hence a representation such as our  $\bar{\rho}$  should be obtainable from the Deligne representation  $\rho_k$  attached to forms of weight  $k$  on  $\text{SL}(2, \mathbf{Z})$ . Although our methods “find” in  $\rho_k$  a representation  $\bar{\rho}$  which satisfies the first three properties of (1.3), a proof that this representation satisfies (iv) would seem to require unknown Galois-theoretic properties of étale cohomology. This leads to the second idea of Serre, that (mod  $p$ ) representations coming from  $\rho_k$  ought to be visible (at least up to twist) on the Jacobian variety  $J_1(p)$ . (A similar idea is the starting point in a recent paper of Koike [10].) This is what led us to look at forms of weight 2.

We hope that our method will apply also to more general Kummer-like criteria, such as that given by Greenberg [7]. Some relevant computations have been made by Yamauchi [21].

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## § 2. Reductions of Reducible Representations

Let  $K$  be a finite extension of  $\mathbf{Q}_p$ . Let  $\mathcal{O}$  be its integer ring,  $\mathbf{F}$  the residue field, and  $\pi$  a uniformizing parameter. Let  $V$  be a free module of rank 2 over  $K$ . A lattice in  $V$  is a free  $\mathcal{O}$ -module of rank 2 in  $V$  which generates  $V$  over  $K$ .

We suppose given a representation

$$\rho: G \rightarrow \mathbf{GL}(V)$$

of a group in  $V$  such that  $G$  leaves stable *some* lattices of  $V$ . (This latter condition is always satisfied if  $G$  is compact and  $\rho$  is continuous, for example.) If  $T \subset V$  is stable by  $G$ ; then  $G$  acts on  $T/\pi T$ , which is free of rank 2 over  $F$ . The associated map

$$\bar{\rho}: G \rightarrow \mathbf{GL}(T/\pi T)$$

will be called the *reduction* of  $\rho$  attached to  $T$ . It is known that the semi-simplification of  $\bar{\rho}$  (as an  $\mathbf{F}$ -representation) is independent of the choice of  $T$  [4, 30.16], so that  $\bar{\rho}$  is unique if one reduction (and hence every reduction) is simple.

We consider, however, the opposite situation, where the reductions are all reducible. Their semi-simplifications are then described by two characters  $\varphi_1, \varphi_2: G \rightarrow \mathbf{F}^*$ , which do not depend on the choice of  $T$ . A given reduction may be written matrixially in one of the forms:

$$\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}, \quad \begin{pmatrix} \varphi_1 & 0 \\ * & \varphi_2 \end{pmatrix}.$$

It is diagonalizable (i.e., semi-simple) if and only if its image has order prime to  $p$ .

(2.1) **Proposition.** *Suppose that the  $K$ -representation  $\rho$  is simple but that its reductions are reducible. Let  $\varphi_1$  and  $\varphi_2$  be the characters associated to the reductions of  $\rho$ . Then  $G$  leaves stable some lattice  $L \subset V$  for which the associated reduction is of the form  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$  but is not semi-simple.*

*Proof.* Choose a  $G$ -stable lattice of  $V$  together with an  $\mathcal{O}$ -basis of this lattice. Then  $\rho$  may be viewed as a map  $G \rightarrow \mathbf{GL}(2, \mathcal{O})$ . Any matrix  $M \in \mathbf{GL}(2, K)$  such that  $M\rho(G)M^{-1} \subseteq \mathbf{GL}(2, \mathcal{O})$  then defines another  $G$ -stable lattice together with a basis of it. The reduction attached to this new lattice is the map

$$G \rightarrow M\rho(G)M^{-1} \hookrightarrow \mathbf{GL}(2, \mathcal{O}) \rightarrow \mathbf{GL}(2, \mathbf{F}).$$

To prove the proposition, we do some calculations based on the formula

$$P \begin{pmatrix} a & \pi b \\ c & d \end{pmatrix} P^{-1} = \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix},$$

where  $P$  is the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}$ .

We first note that we may assume at the outset that the reduction of the given map  $G \rightarrow \mathbf{GL}(2, \mathcal{O})$  is of the form  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$  rather than the form  $\begin{pmatrix} \varphi_1 & * \\ * & \varphi_2 \end{pmatrix}$ , because if the latter occurs we can divide the upper-right corner entries by  $\pi$  and multiply the lower-left corner entries by  $\pi$  using the formula above. Let us make this assumption together with the following one: each reduction  $\bar{\rho}$  of the form  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$

is semi-simple. With these assumptions, we will show that  $\rho$  is itself reducible, and thus prove (2.1) by contradiction.

Set  $M_0 = I$  ( $2 \times 2$  identity matrix). Inductively, we will define a converging sequence of matrices  $M_i = \begin{pmatrix} 1 & t_i \\ 0 & 1 \end{pmatrix}$  such that  $M_i \rho(G) M_i^{-1}$  consists of elements of  $\mathbf{GL}(2, \mathcal{O})$  whose lower-left corner entries are divisible by  $\pi$  and whose upper-right corner entries are divisible by  $\pi^i$ . This will prove that  $\rho$  is reducible because the matrix  $M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$  with  $t = \text{Lim } t_i$  will then be such that  $M \rho(G) M^{-1}$  consists of matrices whose upper-right corner entries are 0.

According to the conjugation formula above, the induction assumption may be rephrased as follows:  $P^i M_i \rho(G) M_i^{-1} P^{-i}$  consists of integral matrices whose lower-left corner entries are divisible by  $\pi^{i+1}$ . With this assumption, the representation  $\sigma \mapsto P^i M_i \rho(\sigma) M_i^{-1} P^{-i} \pmod{\pi}$  is in the form  $\begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$  because  $\sigma \mapsto \rho(\sigma) \pmod{\pi}$  is of this form. The representation in question is then by assumption semi-simple, so we may choose an element  $u$  of  $\mathcal{O}$  such that the matrix  $U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$  diagonalizes the  $\pmod{\pi}$  representation. That is, we can find a  $u$  in  $\mathcal{O}$  so that

$$U P^i M_i \rho(G) M_i^{-1} P^{-i} U^{-1}$$

consists of matrices whose upper-right corner entries are divisible by  $\pi$  (and whose lower-left corner entries are still divisible by  $\pi^{i+1}$ : conjugation by  $U$  leaves unchanged the lower-left corner of any matrix). This gives that

$$(P^{-i} U P^i M_i) \rho(G) (P^{-i} U P^i M_i)^{-1}$$

consists of integral matrices whose lower-left corner entries are divisible by  $\pi$  and whose upper-right corner entries are divisible by  $\pi^{i+1}$ . Thus we may continue the induction by setting

$$M_{i+1} = P^{-i} U P^i M_i = \begin{pmatrix} 1 & t_i + \pi^i u \\ 0 & 1 \end{pmatrix}.$$

This formula makes visible the fact that  $\{M_i\}$  converges.

### § 3. A Congruence between a Cusp Form and an Eisenstein Series

Let  $p$  be an odd prime and let  $\mu_{p-1}$  be the group of complex  $(p-1)$ -st roots of unity. We consider modular forms of weights 1 and 2 on  $\Gamma_1(p)$ . For a character

$$\varepsilon: (\mathbf{Z}/p\mathbf{Z})^* \rightarrow \mu_{p-1}$$

(possibly the trivial one) we say that a form is of type  $\varepsilon$  if it satisfies the equation

$$f \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \varepsilon(d) \cdot f$$

for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $\Gamma_0(p)$ . (We lift  $\varepsilon$  as usual to a function on  $\mathbf{Z}$ .) A form of type  $\varepsilon$  is a cusp form if its  $q$ -expansion and that of  $f \Big| \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$  both commence with 0; if the  $q$ -expansion of  $f$  commences with 0, then we say that  $f$  is a semi cusp form.

We will have need of the Eisenstein series. Let  $\varepsilon$  be a non-trivial even character. Then the two series

$$G_{2,\varepsilon} = L(-1, \varepsilon)/2 + \sum_{n \geq 1} \sum_{d|n} \varepsilon(d) dq^n,$$

$$s_{2,\varepsilon} = \sum_{n \geq 1} \sum_{d|n} \varepsilon(n/d) dq^n$$

are each of weight 2 and type  $\varepsilon$ . The space of modular forms of weight 2 and type  $\varepsilon$  is generated by the cusp forms and these two series, while the space of semi cusp forms of weight 2 and type  $\varepsilon$  is generated by  $s_{2,\varepsilon}$  and the cusp forms. When  $\varepsilon$  is the trivial character, we still have an Eisenstein series  $G_{2,\varepsilon}$  as above; it may be written

$$\frac{p-1}{24} + \sum_{n \geq 1} \sum_{\substack{d|n \\ p \nmid d}} dq^n.$$

In weight 1 we use the series

$$G_{1,\varepsilon} = L(0, \varepsilon)/2 + \sum_{n \geq 1} \sum_{d|n} \varepsilon(d) q^n$$

when  $\varepsilon$  is an odd character. The Eisenstein series are eigenforms for the Hecke operators  $T(n)$ , at least when  $n$  is prime to  $p$ .

Now fix a prime ideal  $\mathfrak{p}|p$  of the field  $\mathbf{Q}(\mu_{p-1})$ . Then let  $\omega: (\mathbf{Z}/p\mathbf{Z})^* \xrightarrow{\sim} \mu_{p-1}$  be the unique character which satisfies

$$\omega(d) \equiv d \pmod{\mathfrak{p}}$$

for all  $d \in \mathbf{Z}$ .

(3.1) **Lemma.** *Let  $k$  be even,  $2 \leq k \leq p-3$ . Then the modular forms  $G_{2,\omega^{k-2}}$  and  $G_{1,\omega^{k-1}}$  have  $\mathfrak{p}$ -integral  $q$ -expansions in  $\mathbf{Q}(\mu_{p-1})$  which are congruent modulo  $\mathfrak{p}$  to the  $q$ -expansion*

$$-B_k/2k + \sum_{n \geq 1} \sum_{d|n} d^{k-1} q^n.$$

*Proof.* Aside from the constant terms of the series, the assertion follows immediately from the choice of  $\omega$ . To prove the assertions about constant terms, we use the expressions

$$L(0, \varepsilon) = \frac{-1}{p} \sum_{n=1}^{p-1} \varepsilon(n)(n-p/2),$$

$$L(-1, \varepsilon) = \frac{-1}{2p} \sum_{n=1}^{p-1} \varepsilon(n)(n^2 - pn + p^2/6)$$

of the  $L$ -values as generalized Bernoulli numbers, valid for any character  $\varepsilon \pmod{p}$ , cf. [11]. Using the congruence  $\omega(n) \equiv n^p \pmod{p^2}$ , we find

$$pL(0, \omega^{k-1}) \equiv - \sum_{n=1}^{p-1} n^{1+p(k-1)} \pmod{p^2},$$

$$pL(-1, \omega^{k-2}) \equiv \frac{-1}{2} \sum_{n=1}^{p-1} n^{2+p(k-2)} \pmod{p^2}.$$

On the other hand, if  $t$  is a positive even integer we have

$$pB_t \equiv \sum_{n=1}^{p-1} n^t \pmod{p^2}$$

according to [1, (8.8), p. 385]. The desired result follows by combining these facts with the Kummer congruence [1, Th. 5, p. 385].

(3.2) **Corollary.** *Let  $k$  be as above, and let  $n$  and  $m$  be even integers,  $2 \leq n, m \leq p-3$ , satisfying  $n+m \equiv k \pmod{p-1}$ . Then the product*

$$G_{1, \omega^{n-1}} G_{1, \omega^{m-1}}$$

is a modular form of weight 2 and type  $\omega^{k-2}$  whose  $q$ -expansion coefficients are  $p$ -integers in  $\mathbf{Q}(\mu_{p-1})$ . Its constant term is a  $p$ -unit provided that neither  $B_n$  nor  $B_m$  is divisible by  $p$ .

*Proof.* Clear.

(3.3) **Theorem.** *Let  $k$  be as above. Then there exists a modular form  $g$  of weight 2 and type  $\omega^{k-2}$  whose  $q$ -expansion coefficients are  $p$ -integers in  $\mathbf{Q}(\mu_{p-1})$  and whose constant term is 1.*

*Proof.* It suffices to construct a  $g$  whose constant term is a  $p$ -unit. We first try the Eisenstein series  $G_{2, \omega^{k-2}}$ . By (3.1), this form will commence with a unit coefficient unless  $p|B_k$ . If this happens, we then try the products  $G_{1, \omega^{n-1}} G_{1, \omega^{m-1}}$  as in (3.2). If none of these products works, then for every pair  $n, m$  as in (3.2) at least one of the two numbers  $B_n, B_m$  is divisible by  $p$ . Now let  $t$  be the number of even integers  $n, 2 \leq n \leq p-3$ , such that  $p$  divides  $B_n$ . Then elementary reasoning shows that  $t \geq (p-1)/4$  if the theorem is false. However, we have  $p^t | h_p^*$ , where the integer  $h_p^*$  is the so-called first factor of the class number of  $\mathbf{Q}(\mu_p)$  (see below). Hence to prove the theorem it will suffice to prove that

$$h_p^* < p^{(p-1)/4}.$$

According to Carlitz and Olson [3], we may write  $h_p^*$  in the form  $\pm D/p^{(p-3)/2}$ , where  $D$  is a certain determinant of dimension  $(p-1)/2$  whose entries are integers between 1 and  $p-1$ . As Carlitz has pointed out [2], Hadamard's inequality then immediately gives

$$h_p^* < p^{(p+3)/4} 2^{-(p-1)/4}.$$

This implies the desired inequality because  $h_p^* = 1$  for  $p \leq 19$  and  $p \leq 2^{(p-1)/4}$  for  $p > 19$ .

To prove that  $p'$  divides  $h_p^*$  we use the expression

$$h_p^* = \alpha p \prod_{\substack{k=2 \\ k \text{ even}}}^{p-1} L(0, \omega^{k-1}),$$

where  $\alpha$  is a certain power of 2 [7, p. 250]. It will be enough to show that  $p'$  divides  $h_p^*$  since  $p$  is unramified. Now, by the  $L(0, \varepsilon)$  formula given above, the quantity  $p \cdot L(0, \omega^{p-2})$  is an algebraic integer. Thus what we want follows from (3.1): if  $p|B_k$  with  $2 \leq k \leq p-3$ , then  $p$  divides  $L(0, \omega^{k-1})$ .

*Remarks.* 1. Masley and Montgomery [13] give the bounds

$$(2\pi)^{-p/2} p^{(p-25)/4} \leq h_p^* \leq (2\pi)^{-p/2} p^{(p+31)/4}$$

for primes  $p$  bigger than 200. This shows that the elementary upper bound for  $h_p^*$  that we use is in fact reasonably sharp.

2. Theorem (3.3) may be proved more conceptually by methods of Mazur [14], using the Deligne-Rapoport study of the modular curve  $X_1(p)$  at the prime  $p$  [5, p. DeRa-108]. One sees by Mazur's technique that  $g$  may be chosen so as to vanish at the cusp 0 of  $X_1(p)$ .

From this point on, we fix an even integer  $k$  ( $2 \leq k \leq p-3$ ) and make the assumption that  $p|B_k$ . We put  $\varepsilon = \omega^{k-2}$ . Since  $B_2 = 1/6$ ,  $k$  is in fact at least 4; hence  $\varepsilon$  is a non-trivial even character. All modular forms will now be of weight 2 and type  $\varepsilon$ .

(3.4) **Proposition.** *There exists a semi cusp form  $f = \sum_{n \geq 1} a_n q^n$  such that the  $a_n$  are  $p$ -integers in  $\mathbf{Q}(\mu_{p-1})$  and such that*

$$f \equiv G_k \equiv G_{2,\varepsilon} \pmod{p}$$

*in  $q$ -expansions.*

*Proof.* Take  $f = G_{2,\varepsilon} - c \cdot g$ , where  $c$  is the constant term of  $G_{2,\varepsilon}$ . Then  $f$  is a semi cusp form by construction, and we have  $f \equiv G_{2,\varepsilon}$  because  $p|c$  by (3.1) and the assumption  $p|B_k$ . Also  $G_{2,\varepsilon} \equiv G_k$  by (3.1).

(3.5) **Proposition.** *There exists a non-zero cusp form  $f'$  of type  $\varepsilon$  which is an eigenform for all Hecke operators  $T_n$  with  $(n, p) = 1$  and which has the property that for each prime  $l \neq p$  the eigenvalue  $\lambda(l)$  of  $T(l)$  acting on  $f'$  satisfies*

$$\lambda(l) \equiv 1 + l^{k-1} \equiv 1 + \varepsilon(l)l \pmod{\mathcal{M}},$$

where  $\mathcal{M}$  is a certain prime (independent of  $l$ ) lying over  $p$  in the field  $\mathbf{Q}(\mu_{p-1}; \lambda(n))$  generated by the eigenvalues over  $\mathbf{Q}(\mu_{p-1})$ .

*Proof* (cf. Koike [9]). The semi cusp form  $f$  of (3.4) is a mod  $p$ -eigenform for the Hecke operators, because it is congruent to the eigenform  $G_{2,\varepsilon}$ . Its mod  $p$ -eigenvalues are congruent to those desired of  $f'$ . Hence we can apply the Deligne-Serre lemma [6, 6.1.1] to get a semi cusp form  $f'$  as in the statement of the proposition. We then must show that this  $f'$  is in fact a cusp form. But as remarked above, the space of semi cusp forms is generated by the space of cusp forms and the eigenform  $s_{2,\varepsilon}$ . Hence it suffices to show that  $f'$  cannot be  $s_{2,\varepsilon}$ . However the eigen-

value of  $T(l)$  acting on  $s_{2,\varepsilon}$  is  $\varepsilon(l) + l$ , and it is clear that we cannot have

$$\varepsilon(l) + l \equiv 1 + l\varepsilon(l) \pmod{p}$$

unless  $\varepsilon(l) = 1$ . Since  $\varepsilon$  is a non-trivial character, this gives what is wanted.

(3.6) **Proposition.** *Any form  $f'$  as in (3.5) is an eigenform for all Hecke operators  $T(n)$  (including those for which  $p|n$ ). Hence, after replacing  $f'$  by a multiple of  $f'$ , we have*

$$f' = \sum_{n=1}^{\infty} \lambda(n) q^n$$

with  $f' | T(n) = \lambda(n) f'$ .

*Proof.* This follows directly from (3.5) and the theory of newforms (see, e.g., [12, Th. 3]) since there are no non-zero forms of weight 2 on  $\mathbf{SL}(2, \mathbf{Z})$ .

We restate what we have concluded from the hypothesis  $p|B_k$ :

(3.7) **Theorem.** *There exists a cusp form  $f = \sum_{n \geq 1} a_n q^n$  of weight 2 and some type  $\varepsilon$  which is a normalized ( $a_1 = 1$ ) eigenform for all Hecke operators  $T(n)$  and which satisfies*

$$a_l \equiv 1 + l^{k-1} \equiv 1 + \varepsilon(l)l \pmod{p}$$

for all primes  $l \neq p$ , where  $\mathfrak{p}$  is a certain prime ideal over  $p$  in the field  $K$  generated by the coefficients of  $f$ , which does not depend on  $l$ .

Note that we may view  $\varepsilon$  as a (non-trivial) character with values in  $K^*$ , since formulas for the Hecke operators show that the values of  $\varepsilon$  lie in the field generated by the coefficients of  $f$ .

#### § 4. Construction and Study of the (mod $\mathfrak{p}$ ) Representation

We retain the notations  $f, \mathfrak{p}, K$  of (3.7). In addition, we let  $\mathcal{O}$  be the integer ring of  $K$ ,  $\mathcal{O}_{\mathfrak{p}}$  its completion at  $\mathfrak{p}$ ,  $K_{\mathfrak{p}}$  the completion of  $K$  at  $\mathfrak{p}$ ,  $\mathbf{F}$  the residue field of  $\mathcal{O}_{\mathfrak{p}}$ ,  $\pi \in \mathcal{O}_{\mathfrak{p}}$  a uniformizing parameter.

We let  $A/\mathbb{Q}$  be the abelian variety attached to  $f$  by Shimura's construction [18, Th. 7.14]. We recall the following properties of  $A$ :

(i) The dimension of  $A$  is equal to the integer  $[K:\mathbb{Q}]$ , and  $K$  is included as a subring of the  $\mathbb{Q}$ -algebra  $(\text{End}_{\mathbb{Q}} A) \otimes \mathbb{Q}$  of endomorphisms of  $A$  defined over  $\mathbb{Q}$ . Thus the  $\mathfrak{p}$ -adic Tate module

$$V_{\mathfrak{p}} = V_p(A) \otimes_{K \otimes \mathbb{Q}_p} K_p$$

is a free  $K_p$ -module of rank 2 on which  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  acts.

(ii) The variety  $A$  is a factor (over  $\mathbb{Q}$ ) of the quotient of the modular variety  $J_1(p)$  by the image in  $J_1(p)$  of the variety  $J_0(p)$ . In particular,  $A$  has good reduction at all primes  $l \neq p$  so that  $V_{\mathfrak{p}}$  is unramified at all such primes. Furthermore, by a theorem of Deligne-Rapoport [5, Ex. 3.7(i), p. DeRa-113],  $A$  acquires everywhere good reduction over the real cyclotomic field  $\mathbb{Q}(\mu_p)^+$ .

(iii) (Eichler-Shimura relation [19, Th. 1.4]). If  $F_l \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is a Frobenius element for a prime  $l \neq p$ , then the trace (resp., determinant) of its action on the  $K_p$ -vector space  $V_p$  is  $a_l$  (resp.,  $l \cdot \varepsilon(l)$ ), regarded as an element of  $K_p$ .

Now we let  $\rho: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}_{K_p} V_p$  be the map arising from the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on  $V_p$ . From (iii) we deduce that the determinant of  $\rho$  is the product  $\chi \varepsilon$ , where we now regard  $\varepsilon$  as a character of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  and where  $\chi$  is the standard cyclotomic character

$$\chi: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbf{Z}_p^* \subseteq K_p^*.$$

(4.1) **Proposition.** *The  $K_p$  representation  $\rho$  is irreducible.*

*Proof.* Suppose otherwise. Then the semi-simplification of  $\rho$ , which is abelian, is described by two characters  $\rho_1, \rho_2: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow K_p^*$ . It is locally algebraic by [17, p. III-20] (or else because it comes from an abelian variety), so that each  $\rho_i$  may be written as an integral power  $\chi^{n_i}$  of  $\chi$  on an open subgroup of an inertia group for  $p$  in  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . This implies that  $\rho_i = \chi^{n_i} \varepsilon_i$ , where  $\varepsilon_i$  is a character of finite order ramified only at  $p$ . Regarding the  $\varepsilon_i$  as Dirichlet characters, we have (for  $l \neq p$ ) the equations

$$l^{n_1 + n_2} \varepsilon_1(l) \varepsilon_2(l) = l \varepsilon(l),$$

$$a_l = \varepsilon_1(l) l^{n_1} + \varepsilon_2(l) l^{n_2}$$

because of (iii). From the first equation we get  $n_1 + n_2 = 1$ , so that one of the  $n_i$ , say  $n_1$ , is at least 1. Therefore  $n_2 \leq 0$ . Looking at the second equation, we now see that  $|a_l| \geq l - 1$  for all  $l \neq p$ . When  $l \geq 7$ , however, this contradicts the ‘‘Riemann hypothesis’’  $|a_l| \leq 2\sqrt{l}$ .

From now on, we use  $\chi$  to denote the character ‘‘ $\chi \pmod p$ ,’’ namely the composition

$$\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\chi} \mathbf{Z}_p^* \rightarrow \mathbf{F}_p^* \hookrightarrow \mathbf{F}^* .^1$$

(4.2) **Proposition.** *There exists an  $\mathcal{O}_p$ -lattice  $L \subset V_p$  invariant by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  for which the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on  $L/\pi L$  may be described matrixially by*

$$\begin{pmatrix} 1 & * \\ 0 & \chi^{k-1} \end{pmatrix}$$

and is furthermore not semi-simple.

*Proof.* In view of (4.1) and (2.1) it suffices to show that there exists a lattice  $T \subset V_p$  stable by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  for which the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on  $T/\pi T$  is reducible in such a way that its semi-simplification is given by the two characters 1 and  $\chi^{k-1}$ . In fact, let  $T$  be any  $\mathcal{O}_p$ -lattice stable by  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ . By the Eichler-Shimura relation, if  $l \neq p$  then a Frobenius element for  $l$  acts on  $T/\pi T$  with trace  $a_l \pmod{\pi}$  and determinant  $l \varepsilon(l) \pmod{\pi}$ . Because of (3.7) these numbers are respectively congruent to  $l^{k-1} + 1$  and  $l^{k-1} \pmod{\pi}$ . By the Čebotarev Density Theorem, the trace and determinant of the action of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  on  $T/\pi T$  are respectively  $1 + \chi^{k-1}$  and  $\chi^{k-1}$ .

<sup>1</sup> Thus we return to the notation used in the Introduction

According to the Brauer-Nesbitt Theorem [4, Th. 30.16], this implies the desired assertion about  $T/\pi T$ .

Let us set  $M = L/\pi L$ . This will be the representation space for the  $\bar{\rho}$  of (1.3). In fact, property (ii) of this § together with (4.2) shows that the first three conditions of (1.3) are satisfied by the representation. It remains only to verify the fourth condition.

We consider the subgroup  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}(\mu_p)^+)$  of  $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$  corresponding to the real cyclotomic field  $\mathbf{Q}(\mu_p)^+$ . In this subgroup we consider a decomposition group  $D$  for the unique prime of  $\mathbf{Q}(\mu_p)^+$  lying over  $p$ . Since  $p \nmid \chi[\mathbf{Q}(\mu_p)^+ : \mathbf{Q}]$ , to verify the last condition of (1.3) it suffices to prove that the action of  $D$  on  $M$  is semi-simple, i.e. that the image of  $D$  in  $\text{Aut } M$  has order prime to  $p$ . It will be convenient to let  $E$  be the completion of the real cyclotomic field at  $p$  and to identify  $D$  with  $\text{Gal}(\bar{E}/E)$ .

(4.3) **Proposition.** *The  $\text{Gal}(\bar{E}/E)$ -module  $M$  is the Galois module attached to a finite flat commutative group scheme of type  $(p, \dots, p)$  over the integer ring  $\mathcal{R}$  of  $E$ .*

*Proof.* After changing  $A$  by a  $\mathbf{Q}$ -isogeny we may assume that  $\mathcal{O}$  operates on  $A$  and that  $M$  is isomorphic to the “kernel of  $p$ ” on  $A$ . This makes  $M$  isomorphic to a submodule of the module of  $p$ -division points of  $A$ . By the Deligne-Rapoport theorem mentioned above,  $A$  acquires good reduction over  $E$ . Hence the module of  $p$ -division points has the property asserted of  $M$ : it is the Galois module attached to the scheme-theoretic kernel  $\mathcal{A}_p$  of the map “multiplication by  $p$ ” on the Neron model for  $A$  over  $\mathcal{R}$ . Then  $M$  for its part is the Galois module attached to the Zariski closure  $\mathcal{M}$  of  $M$  in  $\mathcal{A}_p$ , cf. [15, § 2].

Before completing the proof that  $M$  is semi-simple as a  $D$ -module, we summarize the properties of  $M$  that we will use:

- (a) It is free of rank 2 over  $\mathbf{F}$ ,
- (b)  $D$  acts trivially on a 1-dimensional subspace  $X$  of  $M$  and via the character  $\chi (= \chi^{k-1})$  on the quotient  $Y = M/X$ .
- (c)  $M$  is the module attached to a finite flat group scheme  $\mathcal{M}$  of type  $(p, \dots, p)$  over  $\mathcal{R}$ .

(4.4) **Theorem.** *The image of  $D$  in  $\text{Aut } M$  has prime-to- $p$  order.*

*Proof.* Let  $\mathcal{X}$  be the Zariski closure of  $X$  in  $\mathcal{M}$ . The  $D$ -module attached to  $\mathcal{X}$  is the trivial module  $X$ , and the absolute ramification index of  $E$  is  $(p-1)/2 < p-1$ . Hence  $\mathcal{X}$  is a non-zero constant group scheme over  $\mathcal{R}$  by the classification theorem of Raynaud [15, Th. (3.3.3)]. Hence  $\mathcal{M}$  cannot be connected, since it has the étale subgroup  $\mathcal{X}$ .

Take the canonical exact sequence of  $D$ -modules

$$0 \rightarrow M^0 \rightarrow M \rightarrow M^{\text{ét}} \rightarrow 0,$$

where  $M^0$  is associated with the largest connected subgroup of  $\mathcal{M}$  and  $M^{\text{ét}}$  with the largest étale quotient. Because  $M$  has a Galois-compatible  $\mathbf{F}$ -vector space structure,  $\mathcal{M}$  is a “group scheme in  $\mathbf{F}$ -vector spaces” by the theorem of Raynaud mentioned above. In particular, the above exact sequence is a sequence of  $\mathbf{F}$ -vector spaces.

Now  $M^0$  is not all of  $M$  because  $\mathcal{M}$  is not connected. And  $M^0 \neq 0$  because  $M^{\text{et}}$  is unramified but  $M$  is not (since it has the quotient  $Y$ ). Thus  $M^0$  is 1-dimensional. Further the fact that  $M^{\text{et}}$  is unramified and  $Y$  isn't shows that the image of  $M^0$  in  $M$  is distinct from  $X$ . Hence  $D$  leaves stable both  $X$  and a line in  $M$  which is distinct from  $X$ . Since any element of order  $p$  in  $\text{Aut } M$  leaves stable a *unique* line, this proves what is wanted.

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